

# On Multiple Schramm-Loewner Evolutions

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## Abstract

In this note we consider the ansatz for Multiple Schramm-Loewner Evolutions (SLEs) proposed by Bauer, Bernard and Kytölä from a more probabilistic point of view. Here we show their ansatz is a consequence of conformal invariance, reparameterisation invariance and a notion of absolute continuity. In so doing we demonstrate that it is only consistent to grow multiple SLEs if their  $\kappa$  parameters are related by  $\kappa_i = \kappa_j$  or  $\kappa_i = \frac{16}{\kappa_j}$ .

## 1 Introduction

Schramm (or Stochastic) Loewner Evolutions (SLEs) are a powerful tool to describe the continuum limit of two-dimensional interfaces in statistical mechanics at criticality [1, 2]\*. Since such statistical mechanic models are also expected to have a conformal field theory interpretation, we are naturally led to understand the connections between conformal field theory (CFT) and SLE [6, 7].

In the case where configurations involve just one interface, the connections are well understood largely due to the work of Bauer and Bernard [6, 8, 9]. However, to understand if there is a role for such CFT notions as fusion and conformal blocks it is necessary to consider configurations involving many interfaces [10, 11].

The question what is the correct SLE description of multiple interfaces consistent with conformal symmetry has been addressed in a paper by Bauer, Bernard and Kytölä (BBK) [12] wherein the authors couple CFTs to multiple SLEs and find the conditions for certain objects to be martingales. From this they provide an ansatz conjectured to describe multiple SLEs consistent with conformal invariance. While this procedure is justified from the statistical mechanic point of view, it is not clear how to interpret elements of their ansatz in probability theory.

In this note we consider multiple SLEs from a more probabilistic point of view. By assuming conformal invariance, reparameterisation invariance of the curves and a notion of absolute continuity we rederive the BBK ansatz. Along the way we demonstrate that it is only consistent to grow multiple SLEs if their  $\kappa$  parameters are related by  $\kappa_i = \kappa_j$  or  $\kappa_i = \frac{16}{\kappa_j}$ , hence we find realisations of both the conformal field theory fields  $\phi_{1,2}$  and  $\phi_{2,1}$ : all the building blocks needed to create general fields  $\phi_{r,s}$  from the Kacs table. This condition can be restated as saying multiple SLEs can only be grown consistently if they all have the same central charge.

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\*For an introduction to this field we suggest [3, 4]. There was also a useful preprint by Lawler which is now a book [5].

The plan of the paper is as follows: First we introduce multiple SLEs and the ansatz of BBK, pointing out some of the questions answered in later sections. We then discuss an application of absolute continuity by looking at the connection between single and multiple SLEs. Section 4 studies the requirement of conformal invariance on multiple SLEs, while section 5 considers reparameterisation invariance following the work of Dubédat [11]<sup>†</sup>. In section 6 we reconsider reparameterisation invariance using a different technique and observe the same results. We end with our conclusions.

## 2 Multiple SLEs and the Ansatz of Bauer, Bernard and Kytölä

The primary object of study in this paper is a space of  $m$  curves in the closure of the upper half plane (UHP) that start and end on the real line plus infinity. These curves may have self intersections and mutual intersections but no crossings. For our purposes it is more helpful to think of  $n$ -curves starting on the real axis at points  $x_0^i$ :  $2m-n$  of these curves go to infinity while the remaining  $n-m$  pair-up to form the total of  $m$  complete curves.

To study this space and probability measures upon it we follow the idea of Schramm [1] to use Loewner evolutions to formulate an equivalent description of the curves as real valued functions. Then the measures on curves lift to measures on a space of real valued functions. The goal of this paper is to study how properties of the curve measure (such as conformal invariance and reparameterisation invariance) translate into properties of the measure on driving functions.

The Loewner evolution map for multiple curves is the solution to,

$$\dot{G}_t(z) = \sum_{i=1}^n \frac{2a_t^i}{G_t(z) - x_t^i}, \quad G_0(z) = z. \quad (2.1)$$

This equation is well defined up to some explosion time  $\tau_z = \inf\{t : G_t(z) \in \{x_t^1, \dots, x_t^n\}\}$ . The set  $K_t = \{z \in \mathbb{H} : \tau_z \leq t\}$  is called the hull and is such that  $G_t : \mathbb{H}/K_t \rightarrow \mathbb{H}$ ,  $\mathbb{H}$  denoting the upper half plane. One can recover the curves  $\gamma_t^i$  from the  $x_t^i$  and  $a_t^i$  via  $\gamma_t^i = \lim_{\varepsilon \rightarrow 0} G_t^{-1}(x_t^i + i\varepsilon)$ , then the hull is the component of the set  $\mathbb{H}/\cup_i \gamma_t^i$  connected to infinity: here we use the notation  $\gamma_t^i$  to denote both the location of the tip of the curve  $i$  at time  $t$  and the set  $\{\gamma_s^i : s \leq t\}$ , we do not expect any confusion.

There are of course many maps which conformally map  $\mathbb{H}/K_t \rightarrow \mathbb{H}$ . However, as a solution to (2.1),  $G_t$  is automatically hydrodynamically normalised, that is to say it is the unique conformal map  $\mathbb{H}/K_t \rightarrow \mathbb{H}$  such that  $\lim_{z \rightarrow \infty} G_t(z) - z = 0$ . In this paper we will meet a lot of conformal maps mapping  $\mathbb{H}/A \rightarrow \mathbb{H}$  for some set  $A \subset \mathbb{H}$ , it will always be implicit that these maps take the component of  $\mathbb{H}/A$  connected to infinity onto  $\mathbb{H}$  and are hydrodynamically normalised.

A natural quantity in Loewner evolutions is the upper half plane capacity,  $\text{hcap}$ , which for a hull  $K_t$  is defined via the expansion,

$$G_t(z) = z + \frac{2\text{hcap}[K_t]}{z} + \mathcal{O}(z^{-2}), \quad (2.2)$$

(note the factor of 2 in this definition) then if the curves  $\gamma_t^i = \gamma_{t^i(t)}^i$  are individually parameterised by  $t^i(t)$ ,  $K_t = K_{t^1(t), \dots, t^n(t)}$  we find from (2.1),

$$a_t^i = \frac{dt^i(t)}{dt} \frac{\partial}{\partial t^i} \text{hcap}[K_{t^1(t), \dots, t^n(t)}]. \quad (2.3)$$

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<sup>†</sup>We note some overlap between the work presented here and the second version of [11].

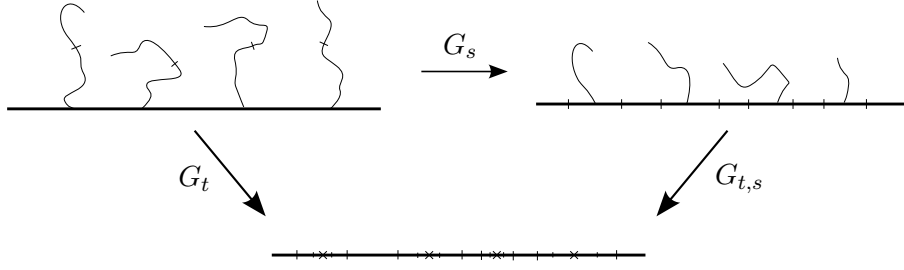


Figure 1 : The maps  $G_t$  and  $G_s$  acting on curves  $\gamma_t^i$ .  
In the left hand diagram, the lines represent the curves up to time  $t$ , while the dashes across the curves represent the positions of the tips at time  $s$ . In the right hand diagram we have used dashes to mark the images of the points  $\gamma_0^i$  under  $G_s$ .

It is important to note that the pair  $(x_t^i, a_t^i)$  encode the curves and their parameterisation: changing the parameterisation changes  $x_t^i$  and  $a_t^i$  in some complicated way.

In the above, we have started with the functions  $(x_t^i, a_t^i)$  and constructed curves with a parameterisation. To go the other way, one takes the maps  $G_t$  associated with the curves and notes that they satisfy an integral equation generalising (2.1) which defines functions  $x_t^i$  and measures  $a_t^i dt = \text{hcap}[K_{t^1(t), \dots, t^i(t+dt), \dots, t^n(t)}]$ . To write (2.1) we must assume some absolute continuity for these measures. We will do this for simplicity.

## 2.1 The Ansatz of Bauer, Bernard and Kytölä

In the case of SLE for a single curve, it was shown by Schramm [1] that conformal invariance implies the driving function  $x_t$  should be a continuous Markov process which, in a particular parameterisation, has independent increments. This requires  $x_t = \sqrt{\kappa} B_t$  for some standard Brownian motion  $\langle B, B \rangle_t = t$ . By a time change, in a general parameterisation  $x_t$  is a continuous martingale with quadratic variation  $\langle x, x \rangle_t = \kappa \text{hcap}[K_t]$ .

For multiple SLEs, the same argument shows the driving functions  $x_t^i$  should be a continuous Markov process. Indeed, consider distribution of  $n$ -multiple SLE curves in the upper half plane and consider two Loewner maps  $G_t$  and  $G_s$ ,  $s < t$  (see figure 1). Note that  $G_{t,s} = G_t \circ G_s^{-1}$  is the solution to,

$$\frac{dG_{t,s}(z)}{dt} = \sum_i \frac{2a_t^i}{G_{t,s}(z) - x_t^i}, \quad G_{s,s}(z) = z. \quad (2.4)$$

By conformal invariance, the distribution of  $G_{t,s}$  given  $x_s^i$  should be the same as a Loewner evolution starting from  $x_s^i$  and parameterised by  $a_t^i$  in the same way. In particular, this distribution should be independent of  $x_r^i$  for  $r < s$ . As the distribution of  $G_{t,s}$  is encoded in the distribution of  $x_t^i$ , this implies the distribution of  $x_t^i$  given  $x_s^i$  should be independent of  $x_r^i$ ,  $r < s$  and so  $x_t^i$  is a Markov process.

However, we need to do more work to extract all the consequences of conformal invariance. In [12], BBK studied multiple SLEs by coupling them to conformal field theory. By arguing that certain CFT quantities should be martingales, they obtained the following stochastic differential equation which the driving functions  $x_t$  consistent with conformal symmetry should satisfy,

$$dx_t^i = dM_t^i + \kappa_i a_t^i \frac{\partial}{\partial x_t^i} \log Z[x_t] dt + \sum_{k \neq i} \frac{2a_t^k}{x_t^i - x_t^k} dt, \quad (2.5)$$

where  $M_t^i$  are independent continuous martingales with quadratic variation  $d\langle M^i, M^j \rangle_t = \kappa_i a_t^i \delta_{ij} dt$ , the function  $Z[x]$  transforms as a tensor under Möbius transformations and it satisfies the “null vector equations”,

$$0 = \frac{\kappa_i}{2} \frac{\partial^2 Z[x]}{\partial x_i^2} - \sum_{k \neq i} \frac{2}{x_i - x_k} \frac{\partial Z[x]}{\partial x_k} - \sum_{k \neq i} \frac{2h_k}{(x_i - x_k)^2} Z[x], \quad (2.6)$$

In their paper, BBK go on to conjecture how the space of solutions to (2.6) represent the different ways of joining the  $n$ -multiple SLEs to create  $m$  simple curves. We refer the reader to BBK for details.

In this note, we will concentrate on understanding the origin of (2.5) and (2.6) without reference to CFT. We will show that the form of (2.5) and the null vector equation follow from conformal and reparameterisation invariance. Some other questions will also be answered:

1. Why does the quadratic variation of the driving martingales have the form  $d\langle M^i, M^j \rangle_t = \kappa a_t^i \delta_{ij} dt$ ? In BBK, they argue that the driving functions should “grow independently of each other on short time scales”, but what does this phrase mean? In the next section we answer this question using a notion of absolute continuity.

2. Do all the martingales have the same  $\kappa$  parameter? In section 5 we will see that this assumption maybe relaxed to  $\kappa_i \in \{\kappa, \frac{16}{\kappa}\}$  and that the restriction is due to reparameterisation invariance.

## 2.2 Girsanov’s theorem

Girsanov’s theorem provides a way of playing with the drift term of stochastic differential equations by changing the probability measure. We will use the theorem in the following situation. Consider a filtration  $\mathcal{F}_t$  and probability measures  $P$  and  $Q$  on  $\mathcal{F}_\infty$  such that the restrictions are absolutely continuous  $Q_t \ll P_t$ . Furthermore, let  $P_t$  and  $Q_t$  be such that the Radon-Nykodym derivative  $D_t$  be continuous and of the form,

$$D_t = \frac{dQ_t}{dP_t} = \exp \left\{ L_t - \frac{1}{2} \langle L, L \rangle_t \right\}, \quad (2.7)$$

for some local martingale  $L_t$ . If  $M_t$  is a continuous  $(\mathcal{F}_t, P)$ -local martingale then,

$$\widetilde{M}_t = M_t - \langle M, L \rangle_t, \quad (2.8)$$

is a continuous  $(\mathcal{F}_t, Q)$ -local martingale. We refer the reader to chapter VIII of [13] for more details. Note that in our application a local martingale  $D_t \geq 0$ , can be a Radon-Nykodym derivative for some change of measure if and only if  $\mathbb{E}[D_t] = 1$ , i.e. it is a true martingale (Proposition VIII.1.13 of [13]).

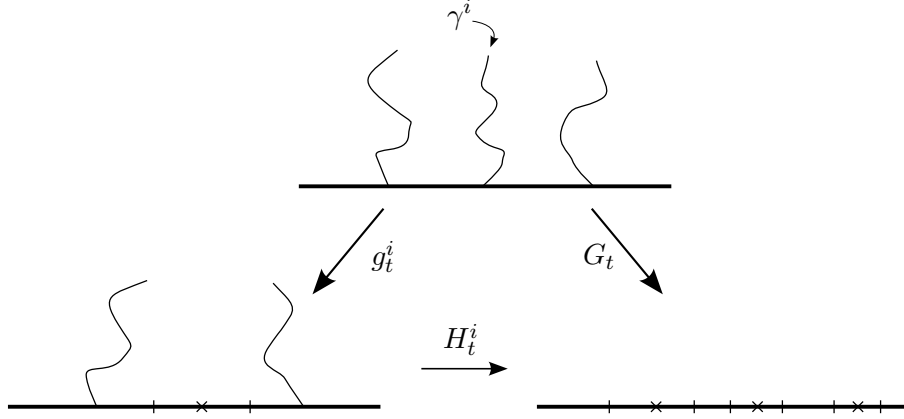


Figure 2 : A diagram representing the map  $H_t^i$ .

### 3 Absolute Continuity

In this section we argue that multiple SLEs which are absolutely continuous wrt a single SLE (up to some stopping time) have the correct quadratic variations for the BBK ansatz. We contend this is the correct notion for being “locally like a single SLE”.

We have seen one natural way to encode multiple SLEs is in terms the single Loewner map  $G_t$ . There is second, useful for situations involving time changes, which uses a different Loewner map for each component curve. In more detail, consider  $n$ -curves  $\gamma_t^i$  each creating its own hull  $K_t^i$  and let these hulls be rectified by maps  $g_t^i : \mathbb{H}/K_t^i \rightarrow \mathbb{H}$  which define Loewner evolutions,

$$\dot{g}_t^i(z) = \frac{2c_t^i}{g_t^i - w_t^i}, \quad g_0^i(z) = z, \quad c_t^i = \frac{d}{dt} \text{hcap}[K_t^i]. \quad (3.9)$$

If all these curves are independent SLEs then the driving functions  $w_t^i$  are independent martingales such that,

$$d\langle w^i, w^j \rangle_t = \kappa_i c_t^i \delta_{ij} dt. \quad (3.10)$$

To relate these driving functions to those of  $G_t$ , it is useful to define the maps  $H_t^i$  (see figure 2),

$$G_t = H_t^i \circ g_t^i. \quad (3.11)$$

It follows from this definition that,

$$x_t^i = H_t^i(w_t^i), \quad a_t^i = H_t^{i'}(w_t^i)^2 c_t^i. \quad (3.12)$$

The first of these relations is trivial. The second requires a little more work which we reproduce from [14]. Consider a small increase in the length of the  $i$ th curve while keeping the others fixed. We need to compare the hcap of a small piece of curve  $\delta\gamma_t^i = g_t^i(\gamma_{t+\delta t}^i)$ ,  $\text{hcap}[\delta\gamma_t^i] \sim c_t^i \delta t$ , with that of the image of this curve under  $H_t^i$ . Assuming the curve  $\gamma_{t+\delta t}^i$  does not intersect any of the other

curves (near the time  $t$ ), the map  $H_t^i$  is analytic near  $\delta\gamma_t^i$  and we can approximate  $\hat{\delta\gamma}_t^i = H_t^i(\delta\gamma_t^i) \sim H_t^i(w_t^i) + H_t^{i'}(w_t^i)(\delta\gamma_t^i - w_t^i)$ . The result then follows from the following easy properties of  $\text{hcap}$ : Let  $g_A : \mathbb{H}/A \rightarrow \mathbb{H}$  for some suitable  $A \subset \mathbb{H}$ ,  $\lambda \geq 0$ ,

$$\text{hcap}[A] = \lim_{z \rightarrow \infty} \frac{1}{2}z (g_A(z) - z) , \quad \text{hcap}[\lambda A] = \lambda^2 \text{hcap}[A] . \quad (3.13)$$

Taking the time derivative of (3.11) we obtain,

$$\dot{H}_t(z) = \sum_j \frac{2a_t^j}{H_t^i(z) - x_t^j} - H_t^{i'}(z) \frac{2c_t^i}{z - w_t^i} , \quad (3.14)$$

$$\dot{H}_t(w_t^i) = -3H_t^{i''}(w_t^i)c_t^i + \sum_{k \neq i} \frac{2a_t^k}{x_t^i - x_t^k} \quad (3.15)$$

and hence using Itô's formula,

$$dx_t = H_t^{i'}(w_t^i)dw_t^i + \left(\frac{\kappa_i}{2} - 3\right) H_t^{i''}(w_t^i)c_t^i dt + \sum_{k \neq i} \frac{2a_t^k}{x_t^i - x_t^k} dt \quad (3.16)$$

In particular the quadratic variations are,

$$d\langle x^i, x^j \rangle_t = H_t^{i'}(w_t^i)^2 \kappa_i c_t^i \delta_{ij} dt = \kappa_i a_t^i \delta_{ij} dt . \quad (3.17)$$

Now, it is well known that under absolutely continuous changes of measure the quadratic variation of a process will not change (for example [13]). Hence for any (possibly stopped) multiple SLE process absolutely continuous with respect to  $n$ -independent SLEs, the quadratic variation of the driving functions will be given by (3.17).

## 4 Conformal Invariance

We assume the driving functions satisfy a stochastic differential equation of the form,

$$dx_t^i = dM_t^i + \sum_j Q_t^{ij}[x_t] a_t^j dt , \quad (4.18)$$

where  $M_t^i$  is a martingale with quadratic variation  $d\langle M^i, M^i \rangle_t = \kappa_i a_t^i dt$ .

To see the effect of a conformal transformation on the form of these equations we follow the arguments of [14], generalised to our situation. Consider a Möbius transformation  $h$  of the UHP to itself. This map takes the curves  $\gamma_t^i$  to new curves  $\hat{\gamma}_t^i = h(\gamma_t^i)$ , which may be encoded by a Loewner evolution with a new set of driving functions<sup>‡</sup>,

$$\dot{\hat{G}}_t = \sum_i \frac{2\hat{a}_t^i}{\hat{G}_t - \hat{x}_t^i} , \quad \hat{G}_0(z) = z . \quad (4.19)$$

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<sup>‡</sup>We avoid the pathological case  $h(\gamma_0^i) = \infty$ .

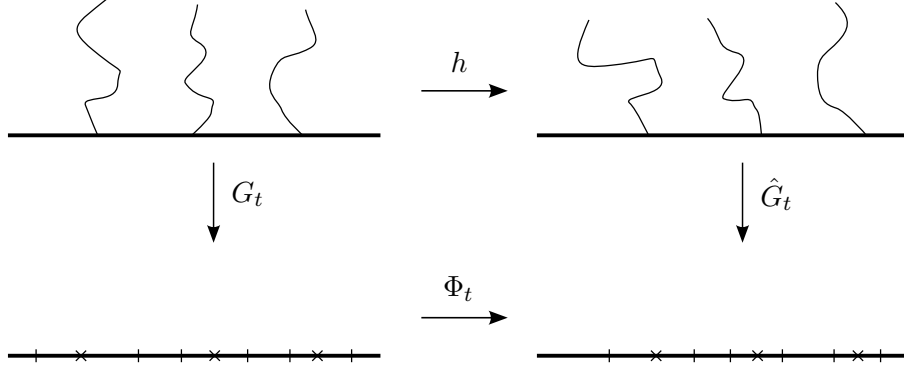


Figure 3 : A diagram representing the map  $\Phi_t$ .

Let  $T = \inf\{t : h(\infty) \in \hat{K}_t\}$  where  $\hat{K}_t$  is the hull generated by  $\cup_i \hat{\gamma}_t^i$ . For times  $t < T$ , the relation between the new driving functions and the old is encoded in the Möbius map  $\Phi_t$  (see figure 3),

$$\Phi_t := \hat{G}_t \circ h \circ G_t^{-1} , \quad (4.20)$$

$$\hat{x}_t^i = \Phi_t(x_t^i) , \quad \hat{a}_t^i = \Phi_t'(x_t^i)^2 a_t^i . \quad (4.21)$$

To obtain the second line we use an identical argument to that for equation (3.12). It follows from (4.20) that,

$$\dot{\Phi}_t(z) = \sum_i \left[ \frac{2\hat{a}_t^i}{\Phi_t(z) - \hat{x}_t^i} - \Phi_t'(z) \frac{2a_t^i}{z - x_t^i} \right] , \quad (4.22)$$

and hence that,

$$\dot{\Phi}_t(x_t^i) = -3\Phi_t''(x_t^i)a_t^i + \sum_{k \neq i} \left[ \frac{2\hat{a}_t^k}{\hat{x}_t^i - \hat{x}_t^k} - \Phi_t'(x_t^i) \frac{2a_t^k}{x_t^i - x_t^k} \right] . \quad (4.23)$$

Using these formulae we apply Itô's formula to  $\hat{x}_t^i$ ,

$$d\hat{x}_t^i = \Phi_t'(x_t^i) dM_t^i + \left[ \sum_{k \neq i} \frac{2\hat{a}_t^k}{\hat{x}_t^i - \hat{x}_t^k} + \left(\frac{\kappa_i}{2} - 3\right) \Phi_t''(x_t^i) a_t^i + \Phi_t'(x_t^i) \left( \sum_j Q_t^{ij}[x_t] a_t^j - \sum_{k \neq i} \frac{2a_t^k}{x_t^i - x_t^k} \right) \right] dt . \quad (4.24)$$

For conformal invariance this new stochastic differential equation should have the same form as the original,

$$d\hat{x}_t^i = d\hat{M}_t^i + \sum_j Q_t^{ij}[\hat{x}_t] \hat{a}_t^j dt . \quad (4.25)$$

with the same functions  $Q_t^{ij}$ . Equating the martingale parts we see,

$$d\hat{M}_t^i = \Phi'_t(x_t^i) dM_t^i, \quad (4.26)$$

and hence  $d\langle \hat{M}^i, \hat{M}^i \rangle_t = \kappa_i \Phi'_t(x_t^i)^2 a_t^i dt = \kappa_i \hat{a}_t^i dt$  as required. Equating the terms of finite variation we find,

$$\sum_j Q_t^{ij}[\hat{x}_t] \hat{a}_t^j - \sum_{k \neq i} \frac{2\hat{a}_t^k}{\hat{x}_t^i - \hat{x}_t^k} = \left(\frac{\kappa_i}{2} - 3\right) \Phi''_t(x_t^i) a_t^i + \Phi'_t(x_t^i) \left( \sum_j Q_t^{ij}[x_t] a_t^j - \sum_{k \neq i} \frac{2a_t^k}{x_t^i - x_t^k} \right). \quad (4.27)$$

Writing

$$P_t^{ii}[x_t] = Q_t^{ii}[x_t], \quad P_t^{ij}[x_t] = Q_t^{ij}[x_t] - \frac{2}{x_t^i - x_t^j}, \quad (4.28)$$

we find the objects  $P_t^{ij}[x_t]$  transform under Möbius transformations as,

$$P_t^{ii}[\Phi_t(x_t)] = \left(\frac{\kappa_i}{2} - 3\right) \Phi''_t(x_t^i) + \Phi'_t(x_t^i) P_t^{ii}[x_t], \quad P_t^{ij}[\Phi_t(x_t)] = \Phi'_t(x_t^i) P_t^{ij}[x_t]. \quad (4.29)$$

and that conformal invariance requires the driving functions satisfy,

$$dx_t^i = \sqrt{\kappa_i} dM_t^i + \sum_j P_t^{ij}[x_t] a_t^j dt + \sum_{k \neq i} \frac{2a_t^k}{x_t^i - x_t^k} dt. \quad (4.30)$$

In the next section we find reparameterisation invariance places further constraints on  $P_t^{ij}$ .

So far in this section we have considered the functions  $Q_t^{ij}[x_t]$  depending only on the driving positions  $x_t^i$ . In more general situations, for example  $SLE(\underline{\kappa}, \underline{\rho})$  [14, 15, 16], one may wish to let  $Q$  depend on some extra parameters evolving via  $G_t$ ,  $Q_t^{ij}[x_t, y_t]$ , where  $y_t^\ell = G_t(y_0^\ell)$ ,  $\ell = 1, \dots, r$ . One can check that this does not affect our result as  $\hat{y}_t^\ell = \Phi_t(y_t^\ell)$  and the new  $P_t^{ij}$  again satisfying,

$$P_t^{ii}[\Phi_t(x_t), \Phi_t(y_t)] = \left(\frac{\kappa_i}{2} - 3\right) \Phi''_t(x_t^i) + \Phi'_t(x_t^i) P_t^{ii}[x_t, y_t], \quad (4.31)$$

$$P_t^{ij}[\Phi_t(x_t), \Phi_t(y_t)] = \Phi'_t(x_t^i) P_t^{ij}[x_t, y_t]. \quad (4.32)$$

It is important to note that in our definition of conformal invariance we have implicitly assumed none of the curves go to infinity. In recovering this eventuality we obtain a particular example of an  $SLE(\kappa, \rho)$  process with  $r = 1$ ,  $y_t = G_t(\infty) = \infty$  and  $\Phi_t(y_t) = \hat{G}_t(h(\infty))$ . Another generalisation involving Lie groups following [17] will be considered in future work.

## 5 Reparameterisation Invariance

The consequences of reparameterisation invariance for multiple SLEs was first considered by Dubédat in [11, 18] wherein a set of equations that are necessary for the driving function of an SLE to be reparameterisation invariant were derived. For completeness, we will rederive these equations and apply



them to our conformally invariant evolution. In so doing we will obtain some interesting results previously assumed from conformal field theory.

To begin, we consider the driving function,

$$dx_t^i = dM_t^i + \sum_j Q_{ij}[x_t] a_t^j dt \quad (5.33)$$

Note that such a process implicitly includes the  $SLE(\kappa, \rho)$  process (with  $r=1$ ,  $y_t=G_t(\infty)=\infty$ ) that describes curves going to infinity. As a Markov process it has the infinitesimal generator,

$$\mathcal{L}_t = \sum_i a_t^i \left[ \frac{\kappa^i}{2} \frac{\partial^2}{\partial x_i^2} + \sum_j Q_{ji}[x] \frac{\partial}{\partial x_j} \right] = \sum_i a_t^i \mathcal{D}_i \quad (5.34)$$

To find Dubédat's condition consider two of the multiple SLE curves,  $i$  and  $j$  say, and chose the time parameterisation such that one first grows the curve  $i$  until  $\text{hcap}[K_t^i] = \varepsilon_i$ , one then grows the curve  $j$  until  $\text{hcap}[K_t^j] = \varepsilon_j$ . An equally valid time parameterisation of the resulting system can be obtained by growing  $j$  and then  $i$ . Dubédat's condition arises from equating the expectations of observables obtained by both ways of growing  $i$  and  $j$ .

Let us begin by growing curve  $i$  and then  $j$  such that both  $\varepsilon_i$  and  $\varepsilon_j$  are infinitesimal (they could be equal). Let  $\delta$  be the time at which the curve  $i$  has grown to size  $\varepsilon_i$  and  $2\delta$  be when both curves have finished growing. We will need (using (3.12) and (3.9)),

$$\begin{aligned} a_0^i \delta &= c_0^i \delta = \varepsilon_i, & a_0^k &= 0, k \neq i \\ a_\delta^j \delta &= H_\delta^{j'}(w_\delta^j)^2 c_\delta^j \delta = H_\delta^{j'}(w_\delta^j)^2 \varepsilon_j, & a_\delta^k &= 0, k \neq j \end{aligned} \quad (5.35)$$

where in this case, the map  $H_\delta^j$  is the Loewner map of the curve  $j$  at time  $\delta$  and so is given by,

$$H_\delta^j(z) = z + \frac{2\varepsilon_i}{z - x_0^i} + \mathcal{O}(\varepsilon_i^2), \quad x_0^j = x_\delta^j + \mathcal{O}(\varepsilon_i) = w_\delta^j + \mathcal{O}(\varepsilon_i) \quad (5.36)$$

$$H_\delta^{j'}(w_\delta^j)^2 = 1 - \frac{4\varepsilon_i}{(x_0^j - x_0^i)^2} + \mathcal{O}(\varepsilon_i^2) \quad (5.37)$$

Now consider the expectation of some functional at time  $2\delta$ . Using the infinitesimal generator this can be expanded to second order in  $\varepsilon_i$  and  $\varepsilon_j$ ,

$$\begin{aligned} \mathbb{E}[f(\gamma_{2\delta})|x_0] &= \mathbb{E}[\mathbb{E}[f(\gamma_{2\delta})|x_\delta]|x_0] = \mathbb{E}[(1 + \delta\mathcal{L}_\delta + \frac{\delta^2}{2}\mathcal{L}_\delta^2)f(\gamma_\delta)|x_0] \\ &= (1 + \delta\mathcal{L}_0 + \frac{\delta^2}{2}\mathcal{L}_0^2)(1 + \delta\mathcal{L}_\delta + \frac{\delta^2}{2}\mathcal{L}_\delta^2)\mathbb{E}[f(\gamma_0)|x_0] \\ &= \left[ 1 + \delta a_0^i \mathcal{D}_i + \frac{\delta^2}{2} a_0^i{}^2 \mathcal{D}_i^2 + \dots \right] \left[ 1 + \delta a_\delta^j \mathcal{D}_j + \frac{\delta^2}{2} a_\delta^j{}^2 \mathcal{D}_j^2 + \dots \right] \mathbb{E}[f(\gamma_0)|x_0] \\ &= \left[ 1 + \varepsilon_i \mathcal{D}_i + \varepsilon_j \mathcal{D}_j - \frac{4\varepsilon_i \varepsilon_j}{(x_i - x_j)^2} \mathcal{D}_j + \frac{\varepsilon_i^2}{2} \mathcal{D}_i^2 + \varepsilon_i \varepsilon_j \mathcal{D}_i \mathcal{D}_j + \frac{\varepsilon_j^2}{2} \mathcal{D}_j^2 + \dots \right] \mathbb{E}[f(\gamma_0)|x_0]. \end{aligned} \quad (5.38)$$

Equating this expression with that obtained by growing  $j$  and then  $i$  we obtain Dubédat's commutation relations,

$$[\mathcal{D}_i, \mathcal{D}_j] = \frac{4}{(x_i - x_j)^2} (\mathcal{D}_j - \mathcal{D}_i) \quad (5.39)$$

or in components,

$$\begin{aligned} \sum_n \left[ \kappa^i \frac{\partial Q_{nj}}{\partial x_i} \frac{\partial}{\partial x_i} - \kappa^j \frac{\partial Q_{ni}}{\partial x_j} \frac{\partial}{\partial x_j} + \frac{\kappa^i}{2} \frac{\partial^2 Q_{jn}}{\partial x_i^2} - \frac{\kappa^j}{2} \frac{\partial^2 Q_{ni}}{\partial x_j^2} + \sum_\ell \left( Q_{\ell i} \frac{\partial Q_{nj}}{\partial x_\ell} - Q_{\ell j} \frac{\partial Q_{ni}}{\partial x_\ell} \right) \right] \frac{\partial}{\partial x_n} \\ - \frac{4}{(x_i - x_j)^2} \left[ \frac{\kappa^j}{2} \frac{\partial^2}{\partial x_j^2} - \frac{\kappa^i}{2} \frac{\partial^2}{\partial x_i^2} + \sum_n (Q_{nj} - Q_{ni}) \frac{\partial}{\partial x_n} \right] = 0 \end{aligned} \quad (5.40)$$

Applying these constraints to our conformal evolution, we first consider the constraints arising from the second order terms. Reintroducing  $P^{ij}$  (and assuming  $\kappa^i \neq 0$ ) we find,

$$\frac{\partial P^{nj}[x]}{\partial x_i} = 0 \text{ for } n \neq j, \quad \kappa^i \frac{\partial P^{jj}[x]}{\partial x_i} = \kappa^j \frac{\partial P^{ii}[x]}{\partial x_j}, \quad (5.41)$$

The first of these relations imply that  $P^{ij}[x]$  is a function of  $x_j$  only. However, this is only consistent with the conformal transformation (4.29) if  $P^{ij} = 0$ . The second equation is an integrability condition,

$$P^{ii} = \kappa^i \frac{\partial}{\partial x_i} F[x]. \quad (5.42)$$

Moving to first order we first notice that terms with  $n \neq i$  or  $j$  are now trivially satisfied. In the case  $n = i$  we find after a little algebra,

$$0 = \kappa^i \frac{\partial}{\partial x_i} \left[ -\frac{1}{(x_j - x_i)^2} - \frac{\kappa^j}{2} \left[ \frac{\partial^2 F[x]}{\partial x_j^2} + \left[ \frac{\partial F[x]}{\partial x_j} \right]^2 \right] + \sum_{\ell \neq j} \frac{2}{x_j - x_\ell} \frac{\partial F[x]}{\partial x_\ell} + \frac{1}{\kappa^i} \frac{6}{(x_i - x_j)^2} \right] \quad (5.43)$$

Writing  $F[x] = \log Z[x]$ ,  $h_i = \frac{6-\kappa_i}{2\kappa_i}$  and letting  $i$  range over  $i \neq j$  we see,

$$A_j(x_j) = \sum_{\ell \neq j} \frac{2h_\ell}{(x_j - x_\ell)^2} - \frac{\kappa^j}{2} \frac{1}{Z[x]} \frac{\partial^2 Z[x]}{\partial x_j^2} + \sum_{\ell \neq j} \frac{2}{x_j - x_\ell} \frac{1}{Z[x]} \frac{\partial Z[x]}{\partial x_\ell} \quad (5.44)$$

However, the conformal properties of  $P^{ii}$  imply that under Möbius transformations,

$$Z[\Phi(x)] = Z[x] \prod_i \Phi'(x_i)^{-h_i}, \quad (5.45)$$

and hence scale and translational covariance (for example) require  $A_j(x_j) = 0$ .

In conclusion, we see that conformal invariance together with reparameterisation invariance imply that the driving function for multiple SLEs should have the form,

$$dx_t^i = dM_t^i + \kappa_i \frac{\partial}{\partial x_t^i} \log Z[x_t] dt + \sum_{k \neq i} \frac{2a_t^k}{x_t^i - x_t^k} dt, \quad (5.46)$$

where the function  $Z[x]$  transforms as (5.45) under Möbius transformations and satisfies the following so called null vector equations,

$$0 = \frac{\kappa_i}{2} \frac{\partial^2 Z[x]}{\partial x_i^2} - \sum_{k \neq i} \frac{2}{x_i - x_k} \frac{\partial Z[x]}{\partial x_k} - \sum_{k \neq i} \frac{2h_k}{(x_i - x_k)^2} Z[x], \quad (5.47)$$

This is precisely the proposal made by Bauer, Bernard and Kytölä [12].

We end this section by proving the following simple theorem,

**Proposition:** *There exists a  $\kappa$  such that the only non-trivial solutions to (5.47) have  $\kappa_i \in \{\kappa, \frac{16}{\kappa}\}$  for all  $i$ .*

To prove it, let us write the null vector equation using a differential operator  $\mathcal{O}_i Z[x] = 0$  then,

$$[\mathcal{O}_i, \mathcal{O}_j] - \frac{4}{(x_i - x_j)^2} (\mathcal{O}_j - \mathcal{O}_i) = -\frac{3(\kappa_i - \kappa_j)(16 - \kappa_i \kappa_j)}{\kappa_i \kappa_j (x_i - x_j)^4} \quad (5.48)$$

and  $Z[x]$  can only be a simultaneous solution to equations  $i$  and  $j$  if the proposition holds.

As observed in previous work on SLEs, it is natural to associate a  $\kappa$  SLE with a  $\phi_{1,2}$  field in conformal field theory<sup>§</sup>. In this case an SLE with  $\frac{16}{\kappa}$  is naturally associated with a  $\phi_{2,1}$  field. We see here that it is quite consistent to put both processes together in the same geometry. Furthermore, by looking at the singularities of the solutions to the null vector equations one can study the fusion of conformal operators. In particular, we now have all the building blocks to construct a probability interpretation for the full Kacs table  $\phi_{r,s}$ . We leave more detailed discussion of this to future work.

Also note that what we have shown is that it is not consistent to put two SLEs together unless their  $\kappa$  parameters are related by the proposition. One could say this is the probability theory realisation of an observation from physics that it is not possible to build a conformal field theory using representations from Virasoro algebras with different central charges,  $c = \frac{1}{2\kappa}(6 - \kappa)(3\kappa - 8)$ .

## 6 Time Changes Take Two

So far in this paper we have found it most natural to use the driving functions  $x_t^i$  to describe our multiple SLE processes. When considering time changes however, it is sometimes better to use the  $w_t^i$  as defined in section 3. In this section we will study conformal multiple SLEs by taking  $n$ -independent SLEs with driving functions  $w_t$  and conditioning them to satisfy (5.46). We will do this in two steps: first we will condition the  $n$ -independent SLEs to move in the background of the other SLEs, then we introduce the drift term involving  $Z[x]$ . As a consequence of this procedure, we will be able to study directly the consequences of reparameterisation invariance.

Before we start with the multiple SLEs however, it is helpful to take a moment to reconsider a single SLE and its image under a conformal map. Consider some hull  $A \subset \mathbb{H}$  away from  $w_0$  with map  $h : \mathbb{H}/A \rightarrow \mathbb{H}$ . Let  $\gamma_t, g_t, w_t$  and  $c_t$  be a standard SLE in the UHP as defined in section 3 and

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<sup>§</sup>See [6, 12] for more details on notation and motivation.

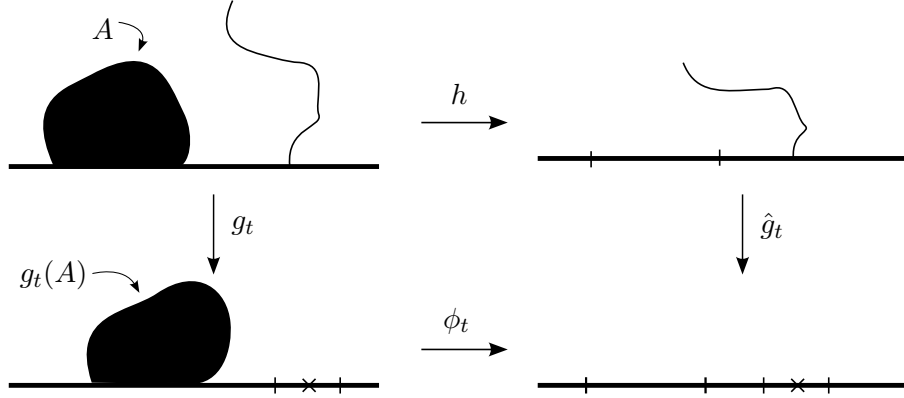


Figure 4 : A diagram representing the map  $\phi_t$ .

define the stopping time  $T = \inf\{t : \gamma_t \in A\}$ . Let  $\hat{\gamma}_t = h(\gamma_t)$ ,  $\hat{g}_t$ ,  $\hat{w}_t$  and  $\hat{c}_t$  be the image of the SLE under the map  $h$ . As in section 4 we use (see figure 4),

$$\phi_t = \hat{g}_t \circ h \circ g_t^{-1}, \quad \hat{w}_t = \phi_t(w_t), \quad \hat{c}_t = \phi'_t(w_t)^2 c_t, \quad (6.49)$$

$$\dot{\phi}_t(w_t) = -3\phi''_t(w_t), \quad \dot{\phi}_t(w_t) = \frac{\phi''_t(w_t)^2}{2\phi'_t(w_t)} - \frac{4\phi'''_t(w_t)}{3}, \quad (6.50)$$

to find that the image driving function satisfies,

$$d\hat{w}_t = \phi'_t(w_t)dw_t + \left(\frac{\kappa}{2} - 3\right) \phi''_t(w_t)c_t dt. \quad (6.51)$$

Now we would like to change the measure such that this new process is an SLE. This is achieved by applying Girsanov's theorem: if we denote the old measure by  $P$ , the required new measure is  $P_{\text{new}} = PD_t$  with Radon-Nykodym derivative<sup>¶</sup>,

$$\begin{aligned} D_t &= \exp \left\{ h \int_0^t \frac{\phi''_s(w_s)}{\phi'_s(w_s)} dw_s - \frac{h^2 \kappa}{2} \int_0^t \frac{\phi''_s(w_s)^2}{\phi'_s(w_s)^2} c_s ds \right\} \\ &= \frac{\phi'_t(w_t)^h}{h'(w_0)^h} \exp \left\{ -\frac{c}{6} \int_0^t S\phi_s(w_s) c_s ds \right\}, \end{aligned} \quad (6.52)$$

wherein  $S$  denotes the Schwarzian derivative,

$$S\phi(z) = \frac{\phi'''(z)}{\phi'(z)} - \frac{3}{2} \frac{\phi''(z)^2}{\phi'(z)^2}, \quad h = \frac{6 - \kappa}{2\kappa}, \quad c = \frac{(6 - \kappa)(3\kappa - 8)}{2\kappa} \quad (6.53)$$

To obtain the second line in (6.52), apply Itô's formula to  $\log \phi'_t(w_t)$  with the help of the formulae in (6.50). Note that (6.52) is the restriction martingale of [14] and  $c$  is the central charge.

<sup>¶</sup>It is easy to check that  $D_t$  is a local martingale, however, to apply Girsanov's theorem we must show that  $D_t$  is a true martingale. We will discuss this at the end of the section.

Under the new measure,  $\hat{w}_t$  is a martingale and  $\hat{\gamma}_t$  is an ordinary SLE in the UHP. By conformal invariance of SLE, the image of this process under the inverse map  $h^{-1}$  is an ordinary SLE in the non-standard domain  $\mathbb{H}/A$ . Hence the process  $w_t$  under the measure  $PD_t$  describes an SLE in the domain  $\mathbb{H}/A$ .

We would like to use this to condition our  $n$ -independent SLEs such that each curve evolves as an SLE living in the background created by the other curves. More precisely, let  $K_t^{i^c}$  be the hull created by all the curves except  $\gamma_t^i$  and let  $h_t^i$  be the map  $h_t^i : \mathbb{H}/K_t^{i^c} \rightarrow \mathbb{H}$ . As usual, let  $g_t^i$  be the Loewner map for the curve  $\gamma_t^i$  and let  $\hat{g}_t^i$  be the Loewner map for the image of  $\gamma_t^i$  under  $h_t^i$ . Following the single curve example we define,

$$H_t^i = \hat{g}_t^i \circ h_t^i \circ g_t^{i-1} \quad (6.54)$$

and note that under an infinitesimal increase in the time along the  $i$ th curve  $t^i$ , while keeping the others fixed, the driving function of the image satisfies,

$$d^i \hat{w}_t^i = H_t^{i'}(w_t^i) dw_t^i + \left(\frac{\kappa_i}{2} - 3\right) H_t^{i''}(w_t^i) c_t^i dt^i, \quad \hat{c}_t^i = H_t^{i'}(w_t^i)^2 c_t^i. \quad (6.55)$$

For this image process to be an SLE we need to change the measure to remove the drift term. By Girsanov's theorem, the Radon-Nykodym derivative is given by,

$$D_t^i = \exp \left\{ h_i \int_0^t \frac{H_s^{i''}(w_s^i)}{H_s^{i'}(w_s^i)} dw_s^i - \frac{h_i^2 \kappa_i}{2} \int_0^t \frac{H_s^{i''}(w_s^i)^2}{H_s^{i'}(w_s^i)^2} c_s^i ds \right\} \quad (6.56)$$

and so the process  $w_t^i$  with the measure  $PD_t^i$  is an SLE moving in the background of the other curves.

Now we grow all the curves simultaneously. As each curve grows we need to adjust the measure to keep it moving in the background of the others. The new conditioned measure is,

$$P_{\text{new}} = P \prod_i D_t^i = P \prod_i H_t^{i'}(w_t^i)^{h_i} \exp \left\{ -\frac{c_i}{6} \int_0^t S H_s^i(w_s^i) c_s^i ds + \sum_{k \neq i} \int_0^t \frac{2 h_i a_s^k}{(x_s^i - x_s^k)^2} ds \right\}. \quad (6.57)$$

To obtain the RHS, we have used Itô's formula applied to  $\log H_t^{i'}(w_t^i)$ , used equation (3.14) to calculate  $\dot{H}_t^{i'}(w_t^i)$  and written  $x_t^i = H_t^i(w_t^i)$ . There is no denominator because  $H_0^i(z) = z$ . The effect of this new measure on the driving functions for the Loewner map  $G_t$  is particularly striking. From section 3 we recall,

$$G_t = H_t^i \circ g_t^i, \quad (6.58)$$

and the driving functions for the multiple SLE are related to those of the single SLEs by,

$$dx_t^i = H_t^{i'}(w_t^i) dw_t^i + \left(\frac{\kappa_i}{2} - 3\right) H_t^{i''}(w_t^i) c_t^i dt + \sum_{k \neq i} \frac{2 a_t^k}{x_t^i - x_t^k} dt. \quad (6.59)$$

This is under the measure  $P$ . Under the new measure  $P_{\text{new}}$  it is easy to check that,

$$N_t^i = \int_0^t H_s^{i'}(w_s^i) dw_s^i + \left(\frac{\kappa_i}{2} - 3\right) \int_0^t H_s^{i''}(w_s^i) c_s^i ds \quad (6.60)$$

is a martingale with quadratic variation  $d\langle N^i, N^i \rangle_t = \kappa_i a_t^i dt$  and equation (6.59) may be rewritten,

$$dx_t^i = dN_t^i + \sum_{k \neq i} \frac{2a_t^k}{x_t^i - x_t^k} dt. \quad (6.61)$$

This is precisely the system derived by Cardy in [10]. The advantage of our derivation is we know how the new driving functions are related to those of the single SLEs and so it is easier to consider the effects of time changes.

The final step in building our conformally invariant process is adjust the measure to introduce the conformal drift term. Using Girsanov's theorem again, this is achieved with the Radon-Nykodym derivative,

$$C_t = \exp \left\{ \sum_i \int_0^t \frac{\partial}{\partial x_s^i} \log Z[x_s] dN_s^i - \frac{1}{2} \sum_i \int_0^t \left( \frac{\partial}{\partial x_s^i} \log Z[x_s] \right)^2 \kappa_i a_s^i ds \right\} \quad (6.62)$$

$$= \frac{Z[x_t]}{Z[x_0]} \exp \left\{ - \sum_i \int_0^t \frac{1}{Z[x_s]} \left[ \sum_{k \neq i} \frac{2}{x_s^k - x_s^i} \frac{\partial Z[x_s]}{\partial x_s^k} + \frac{\kappa_i}{2} \frac{\partial^2 Z[x_s]}{\partial x_s^{i^2}} \right] a_s^i ds \right\} \quad (6.63)$$

where as usual, we have used Itô's formula on  $\log Z[x_t]$ .

To summarise, we have shown that under the measure  $Q = P_{\text{new}} C_t = P C_t \prod_i D_t^i$ , the processes  $x_t^i$  satisfy,

$$dx_t^i = dM_t^i + \kappa_i \frac{\partial}{\partial x_t^i} \log Z[x_t] a_t^i dt + \sum_{k \neq i} \frac{2a_t^k}{x_t^i - x_t^k} dt \quad (6.64)$$

were the  $M_t^i$  are  $Q$ -martingales.

After all that, let us now return to the question of time changes. Consider the expectation value of some functional of our curves with respect to the conformal multiple SLE measure  $Q$ ,

$$\mathbb{E}_Q[f(\gamma_t)] = \mathbb{E}_P[C_t \prod_i D_t^i f(\gamma_t)]. \quad (6.65)$$

Since the original measure  $P$  is reparameterisation invariant, the expectation of an (invariant) object will be invariant if  $C_t \prod_i D_t^i$  is also invariant. The first factor in each term is fine because it only depends on the endpoint, this leaves the contribution from the exponentials:

$$- \sum_i \frac{c_i}{6} \int_0^t S H_s^i(w_s^i) c_s^i ds + \sum_i \int_0^t \frac{1}{Z[x_s]} \left[ \sum_{k \neq i} \frac{2 h_k Z[x_s]}{(x_s^k - x_s^i)^2} - \sum_{k \neq i} \frac{2}{x_s^k - x_s^i} \frac{\partial Z[x_s]}{\partial x_s^k} - \frac{\kappa_i}{2} \frac{\partial^2 Z[x_s]}{\partial x_s^{i^2}} \right] a_s^i ds \quad (6.66)$$

Collecting terms in this way it is clear the second integral is invariant if the integrand vanishes. This is the null vector equation for  $Z[x]$ . Turning to the first term, we introduce coordinates on each curve  $t^i(s) = \text{hcap}[K_s^i]$  so that,

$$c_s^i = \frac{dt^i(s)}{ds} \quad (6.67)$$

and the object of interest becomes the integral of a one-form,

$$dR = - \sum_i \frac{c_i}{6} S H_s^i(w_s^i) dt^i \quad (6.68)$$

For invariance this integral should not depend on the integration path in “time space”. This will be true if and only if the form is closed. Noting that,

$$\frac{\partial H_s^i(w_s^i)}{\partial t^k} = \frac{2H_s^{k'}(w_s^k)^2}{x_s^i - x_s^k}, \quad \frac{\partial H_s^{i'}(w_s^i)}{\partial t^k} = \frac{2H_s^{k'}(w_s^k)^2 H_s^{i'}(w_s^i)}{(x_s^i - x_s^k)^2}, \quad \dots \quad (6.69)$$

$$\frac{\partial}{\partial t^k} S H_s^i(w_s^i) = - \frac{12H_s^{k'}(w_s^k)^2 H_s^{i'}(w_s^i)^2}{(x_s^i - x_s^k)^4}, \quad (6.70)$$

the form is closed if and only if,

$$c_i = c_j, \text{ for all } i \text{ and } j. \quad (6.71)$$

Recalling the definition of the central charge  $c_i$ , (6.53), this is true if and only if,

$$\kappa_i = \kappa_j, \quad \text{or} \quad \kappa_i = \frac{16}{\kappa_j}. \quad (6.72)$$

However, we saw in section 5 that this is also a consequence of the null vector equation so the null vector is sufficient for reparameterisation invariance.

So what have we gained? For one, the calculation we have done here is more general than that of section 5 in that it can be generalised to cases where the object  $Z$  does not define a Markov process. This could be useful in generalising SLE techniques to statistical models with non-conformal boundary conditions.

On the other hand, our method here is not yet a rigorous derivation of necessary conditions for reparameterisation invariance, unlike section 5. To use Girsanov’s theorem properly we need Radon-Nykodym derivatives which are true martingales and not just local. Local martingales are true martingales up to some stopping time, and so the above is valid for suitably stopped processes.

Let us look at this directly. Assuming the null vector equation, our RN-derivative is,

$$C_t \prod_i D_t^i = \frac{Z[x_t]}{Z[x_0]} \prod_i H_t^{i'}(w_t^i)^{h_i} \exp \left\{ -\frac{c_i}{6} \int_0^t S H_s^i(w_s^i) c_s^i ds \right\}. \quad (6.73)$$

From [14], we know  $0 \leq H_t^{i'}(w_t^i) \leq 1$ ,  $S H_t^i(w_t^i) \leq 0$  and that  $H_t^{i'}(w_t^i) \rightarrow 0$  as the curve  $\gamma_t^i$  approaches  $K_t^{i^c}$ . This means that our RN-derivative is well defined and a true martingale so long as we stop the process before the curves intersect. Another way of saying this is that the multiple SLE processes are absolutely continuous wrt  $n$ -independent SLEs away from points where the curves collide or intersect. However, it is precisely the points where the curves collide or bounce off each other that we are most interested in.

For  $c_i \leq 0$  and  $h_i \geq 0$ , the product term in (6.73) is bounded. This requires  $\kappa_i \leq \frac{8}{3}$  or  $\kappa_i = 6$ . For the remainder of this section we will concentrate on the range  $\kappa_i = \kappa \leq \frac{8}{3}$ . We will also work with the particular parameterisation  $a_t^i = 1$ . This is useful since with this choice, the hull created by

a single curve cannot enclose the tip of a second curve. Furthermore, if two curves collide, then the hull created by their union will not contain any other pairs of (growing) curves (We demonstrate this in the appendix). Because of these observations, we order our driving functions  $x_t^1 \leq x_t^2 \leq \dots \leq x_t^n$  and note that two curves collide at their tips if and only if  $x_t^i \rightarrow x_t^{i+1}$ . Also note that  $\text{heap}[K_t] = nt$  and hence a curve only reaches the point  $\infty$  in infinite time.

From the application of conformal field theory, the solutions to the null vector equations are well understood [19, 20] and form a finite dimensional vector space. BBK [12] argued that among the vectors in this space are a set which can be identified with the possible topological configurations of curves. To see part of this picture, we recall that solutions (for  $\kappa_i = \kappa \leq \frac{8}{3}$ ) can have two possible singular behaviours as two points come together:

$$Z[x] \sim C_3(x_i - x_{i+1})^{\frac{2}{\kappa}} Z_3[x_1, \dots, x_{i-1}, \frac{1}{2}(x_i + x_{i+1}), x_{i+2}, \dots, x_n], \quad (6.74)$$

$$Z[x] \sim C_1(x_i - x_{i+1})^{-2h} Z_1[x_1, \dots, x_{i-1}, x_{i+2}, \dots, x_n]. \quad (6.75)$$

Here  $C_1$  and  $C_3$  are constants and the function  $Z_3[x]$  is known to satisfy a third order null vector equation and is possibly singular when its arguments come together or go to infinity. The function  $Z_1[x_1, \dots, x_{i-1}, x_{i+2}, \dots, x_n]$  actually satisfies the same type of null vector equations as  $Z$ , but for  $n-2$  variables.

We now return to the RN-derivative. First we will show that with probability one, multiple SLE processes with  $\kappa_i = \kappa \leq \frac{8}{3}$  will only collide at their endpoints. Let  $I_t^i = \inf_{s < t} \{x_s^{i+1} - x_s^i\}$  and let  $I_t = \min_i \{I_t^i\}$ . Now also recall that  $T$  is the first time that two different curves meet and so it follows that in our chosen parameterisation, the event  $\{T < \infty, I_T = 0\}$  signifies that (at least) two curves meet at their tips while for  $\{T < \infty, I_T > 0\}$  two curves will meet away from their endpoints. From [14] we know then that if the curve  $i$  meets another curve at time  $T < \infty$  then  $\lim_{t \rightarrow T} H_t^{i'}(w_t^i) = 0$  and hence from the formula for the RN-derivative  $\mathbb{E}_Q[\mathbb{1}_{\{T < \infty, I_T > 0\}}] = 0$ . In other words, multiple SLE processes can only meet at infinity or at their endpoints.

We now consider the events  $\{T < \infty, I_T = 0\}$ . The behaviour of the RN-derivative will depend on the singular behaviour of the function  $Z[x]$  as two (or more) points come together. Straightaway we see that if  $Z[x]$  has the first type of singular behaviour (6.74), the limit  $x_i \rightarrow x_{i+1}$  is well defined and the probability that the curve  $i$  and  $i+1$  meet is zero. In the case where  $Z[x]$  is of the second type we have to deal with the singularity.

This singularity represents the fact that while two independent SLEs will meet for the first time at their tips with probability zero, we expect this event will have a finite probability for our multiple SLEs. Hence the measure  $Q$  will be singular with respect to the independent measure  $P$  and the singularity indicates this fact. All is not lost as we do expect to be able to use our RN-derivate to define the measure  $Q$  even in this situation, possibly using an extension theorem (Tulcea's for example). Sadly we are unable to perform this analysis.

If one can make sense of  $\lim_{t \rightarrow T} Q_t$ , then it is very natural extend  $Q_t$  to times  $t > T$  as suggested in BBK. Since  $Z[x]$  has the product for (6.75), we can continue the evolution beyond the collision of curves  $i$  and  $i+1$  by setting  $a_t^i = a_t^{i+1} = 0$  and growing  $n-2$  curves using the driving functions derived from  $Z_1[x]$ . All that we have said so far applies to this new system which will be stopped at some time  $T_2$  when (with probability one) the next pair of curves collide at their tips. After iterating this process, the remaining curves will go to infinity.

In the above, we have concentrated on  $\kappa_i \leq \frac{8}{3}$ . We expect much of what we have said to extend in



principle to  $\kappa_i \leq 4$  or with appropriate modifications to  $\kappa_i < 8$ . However at present we do not have enough control over the RN-derivative to say anything concrete in these cases.

## 7 Conclusions

We have studied a number facets of multiple SLE processes. We have shown that using assumptions of absolute continuity, conformal invariance and reparameterisation invariance, one may rederive the result of BBK [12] obtained previously by assuming a connection to conformal field theory.

We have shown that absolute continuity implies that the quadratic variation of the driving martingale for multiple SLEs should be related to that of a single SLE. We also showed how conformal invariance and reparameterisation invariance imply the form of the multiple SLE driving functions proposed by BBK. As a corollary of this analysis we saw that the  $\kappa$  parameters of each component multiple SLE should be related  $\kappa_i = \kappa_j$  or  $\kappa_i = \frac{16}{\kappa_j}$ , or in other words, the component SLEs should have the same central charge. This provides a probability theory realisation of both  $\phi_{1,2}$  and  $\phi_{2,1}$  fields found in conformal field theory. In the final section, we studied reparameterisation invariance by conditioning  $n$ -independent SLEs. Moreover, we used the associated Radon-Nykodym derivative to study properties of multiple SLEs in the regime  $\kappa_i \leq \frac{8}{3}$ .

Although we believe the general picture is clear, as always with probability there is a great deal of devil in the detail, in particular, our arguments involving the RN-derivative are far from complete. As a final point, we have discussed many sufficient conditions. It would be interesting to see how much of this structure is actually necessary.

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## Appendix

In this appendix we discuss the consequences of choosing a curve parameterisation such that  $a_t^i = 1$  for all  $i$ . From equation (3.12) we note,

$$\text{hcap}[K_t] = \int_0^t \sum_{i=1}^n a_s^i ds = nt , \quad (\text{A-1})$$

$$a_t^i = H_t^{i'}(w_t^i)^2 c_t^i = 1 , \quad \text{hcap}[K_t^i] = \int_0^t c_s^i ds = \int_0^t \frac{1}{H_s^{i'}(w_s^i)^2} ds . \quad (\text{A-2})$$

It also follows from (3.12) that,

$$\begin{aligned} \text{hcap}[K_t^i \cup K_t^j] &= \int_0^t \left[ H_{j,s}^{i'}(w_s^i)^2 c_s^i + H_{i,s}^{j'}(w_s^j)^2 c_s^j \right] ds \\ &= \int_0^t \left[ \frac{1}{H_s^{ij'}(H_{j,s}^i(w_s^i))^2} + \frac{1}{H_s^{ij'}(H_{i,s}^j(w_s^j))^2} \right] ds , \end{aligned} \quad (\text{A-3})$$

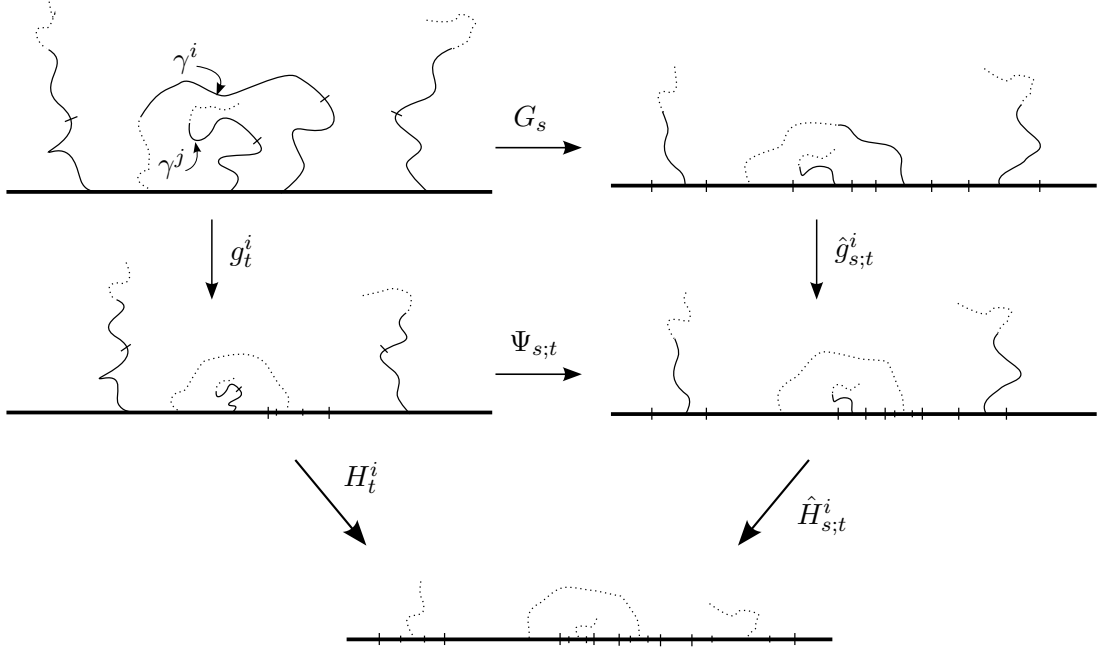


Figure 5 : A diagram to represent the maps  $\hat{H}_{s;t}$  and  $\Psi_{s;t}$ .  
In the top left figure, the solid lines represent the curves up to time  $t$ , the dashes on the curves represent the location of the tips at time  $s$ , and the dotted lines represent the curves from time  $t$  until time  $T$ .

where we have introduced new maps,  $H_{j,t}^i : \mathbb{H}/g_t^i(K_t^j) \rightarrow \mathbb{H}$  and  $H_t^{ij} = H_t^i \circ H_{j,t}^{i-1}$ . The idea behind the notation is as follows: let  $H_{i_1, \dots, i_m, t} : \mathbb{H}/\cup_{r=1}^m K_t^{i_r} \rightarrow \mathbb{H}$  then we define  $H_{j_1 \dots j_p, t}^{i_1 \dots i_m} : \mathbb{H}/H_{i_1, \dots, i_m, t}(\cup_{r=1}^p K_t^{j_r}) \rightarrow \mathbb{H}$ . To reduce indices we also define  $H_t^{i_1, \dots, i_m} = H_{j_1 \dots j_p, t}^{i_1 \dots i_m}$  when the set  $\{j_r\}_{r=1}^p$  contains all indices not included in the set  $\{i_r\}_{r=1}^m$ , in other words  $G_t = H_t^{i_1, \dots, i_m} \circ H_{i_1, \dots, i_m, t}$ . In particular  $H_{i,t} = g_t^i$  and, although we will never use it,  $H_t = H_{1 \dots n, t} = G_t$ .

We will now show that if  $a_t^i = 1$ , then one curve cannot enclose another (ie. cannot disconnect the tip of a curve from infinity). Let us assume this is false and that a curve  $i$  encloses a curve  $j$  for the first time at time  $T$ . Furthermore, let us assume the curve  $i$  does not meet the tip of another curve at time  $T$  (we will deal with this case in a moment) and that the curve  $i$  itself is not enclosed by another curve at time  $T$ . Since  $i$  is enclosing  $j$ , at  $T$  it must hit either the real axis or another curve  $k$  at a point  $\gamma_r^k = \gamma_T^i$  for some time  $r < T$  (possibly  $k = j$ , but not  $k = i$ ). In the second case we may choose a time  $s$ ,  $r < s < T$  and consider the curves  $\hat{\gamma}_t^\ell = G_s(\gamma_t^\ell)$  instead. Hence without loss of generality, we may assume the curve  $\gamma_t^i$  collides with the real axis at  $T$ . Furthermore, (with a suitable choice of  $s$ ) we assume we can arrange that there exists a small neighbourhood around the point  $\gamma_T^i$  which

contains only the curve  $\gamma_T^{\parallel}$ . Under these assumptions, the object  $H_t^{i'}(w_t^i)$  has a well defined limit as  $t \rightarrow T$  and  $H_T^{i'}(w_T^i) > 0$ .

Since  $i$  collides with the real axis,

$$\text{hcap}[K_T^i] = \text{hcap}[K_T^i \cup K_T^j] . \quad (\text{A-4})$$

Moreover because  $0 \leq H_t^{ij'}(z) \leq 1$  we see from (A-3) that  $\text{hcap}[K_T^i \cup K_T^j] \geq 2T$  and hence from (A-2), there exists a time  $t_0$ ,  $0 \leq t_0 \leq T$  such that,

$$\frac{1}{H_{t_0}^{i'}(w_{t_0}^i)^2} \geq \frac{\text{hcap}[K_T^i]}{T} \geq 2 . \quad (\text{A-5})$$

Let us introduce a time  $s$ ,  $0 < s < T$ , and maps  $\hat{H}_{s;j_1, \dots, j_p, t}^{i_1 \dots i_m}$  for  $t, s \leq t \leq T$  defined as  $H$  above but for the sets  $\hat{K}_t^i = G_s(K_t^i)$  (an example of such a map is represented in figure 5). It is not difficult to check that we can also apply our argument to  $\hat{H}_{s;t}^i$  and so obtain that there exists a  $t_1$ ,  $s \leq t_1 \leq T$  such that,

$$\frac{1}{\hat{H}_{s;t_1}^i(w_{t_1}^i)^2} \geq 2 . \quad (\text{A-6})$$

To connect  $\hat{H}_{s;t}^i$  to  $H_t^i$ , we define  $\Psi_{s;t} = \hat{g}_{s;t}^i \circ G_s \circ g_t^{i-1}$  (see figure 5) so  $H_t^i = \hat{H}_{s;t}^i \circ \Psi_{s;t}$ . By setting  $s_n = \frac{1}{2}(T + t_{n-1})$  we construct a sequence  $t_n \rightarrow T$ ,

$$\frac{\Psi'_{s_n; t_n}(w_{t_n}^i)^2}{H_{t_n}^{i'}(w_{t_n}^i)^2} \geq 2 . \quad (\text{A-7})$$

Now  $\lim_{n \rightarrow \infty} H_{t_n}^{i'}(w_{t_n}^i)^2 = \lim_{n \rightarrow \infty} H_{t_n}^{ij'}(H_{j,t_n}^i(w_{t_n}^i))H_{j,t_n}^{i'}(w_{t_n}^i) = H_T^{ij'}(w_T^i) > 0$  using our initial assumptions. On the other hand  $\Psi_{s;t}(z)$  as a function of  $s$  is the multiple Loewner evolution for  $g_t^i(K_s)$  and as such, the Loewner equation shows  $\Psi'_{s;t}(w_t^i)$  is a continuous decreasing function of  $s$ ,  $\Psi'_{s_n; t_n}(w_{t_n}^i) \leq \Psi'_{s_m; t_n}(w_{t_n}^i)$  for  $m < n$  and so,

$$\lim_{n \rightarrow \infty} \Psi'_{s_n; t_n}(w_{t_n}^i) \leq \lim_{n \rightarrow \infty} \Psi'_{s_m; t_n}(w_{t_n}^i) = \Psi'_{s_m; T}(w_T^i) \quad \text{for all } m . \quad (\text{A-8})$$

Since  $\lim_{m \rightarrow \infty} \Psi'_{s_m; T}(w_T^i) = H_T^{ij'}(w_T^i)$  we see the limit of the LHS of (A-7) is bounded above by 1 and we obtain our contradiction.

We can extend these arguments to show that if  $a_t^i = 1$ , two curves joining at the tip cannot enclose one or more (growing) curves. Let us assume that curves  $i$  and  $j$  meet at time  $T$  enclosing a curve  $k$ . Proceeding as before, a formula similar to (A-3) shows that  $\text{hcap}[K_T^i \cup K_T^j \cup K_T^k] \geq 3T$ . However, since  $\lim_{t \rightarrow T} \text{hcap}[K_t^i \cup K_t^j] = \text{hcap}[K_T^i \cup K_T^j \cup K_T^k]$  we know there exists a time  $0 \leq t_0 \leq T$  such that,

$$\frac{1}{H_{t_0}^{ij'}(H_{j,t_0}^i(w_{t_0}^i))^2} + \frac{1}{H_{t_0}^{ij'}(H_{i,t_0}^j(w_{t_0}^j))^2} \geq 3 . \quad (\text{A-9})$$

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<sup>||</sup>This assumption excludes some curves.

This relation is also true for  $\hat{H}_{s;t_1}^{ij}$ ,  $s \leq t_1 \leq T$  and hence by setting  $s_n = \frac{1}{2}(T + t_{n-1})$  we construct a sequence  $t_n \rightarrow T$ ,

$$\begin{aligned} & \frac{1}{\hat{H}_{s_n;t_n}^{ij} (\hat{H}_{s_n;t_n}^i(w_{t_n}^i))^2} + \frac{1}{\hat{H}_{s_n;t_n}^{ij} (\hat{H}_{s_n;t_n}^j(w_{t_n}^j))^2} \\ &= \frac{\Phi'_{s_n;t_n}(H_{j,t_n}^i(w_{t_n}^i))^2}{H_{t_n}^{ij'}(H_{j,t_n}^i(w_{t_n}^i))^2} + \frac{\Phi'_{s_n;t_n}(H_{i,t_n}^j(w_{t_n}^j))^2}{H_{t_n}^{ij'}(H_{i,t_n}^j(w_{t_n}^j))^2} \geq 3, \end{aligned} \quad (\text{A-10})$$

where  $H_t^{ij} = \hat{H}_{s;t}^{ij} \circ \Phi_{s;t}$ . One then proceeds as before to check that  $\lim_{n \rightarrow \infty} \Phi'_{s_n;t_n}(H_{i,t_n}^j(w_{t_n}^j)) \leq \lim_{n \rightarrow \infty} H_{t_n}^{ij'}(H_{i,t_n}^j(w_{t_n}^j))$  and hence the LHS is bounded above by 2 giving the contradiction.

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